Lecture 19

Def - A basis for a subspace $W \subset \mathbb{R}^n$ is a set $B = \{\vec{v}_1, \dots, \vec{v}_k\} \subset W$ such that:

- B is linearly independent
- B spans W

Fund Thm of Lin Alg - Every non-trivial subspace $W \subset \mathbb{R}^n$ has a basis, and any two bases for W have the same # of vectors.

e.g.

$$\mathcal{E} = \{\vec{e}_1, \dots, \vec{e}_n\} \subset \mathbb{R}^n \text{ is a basis for } \mathbb{R}^n$$

Def: The dimension of a subspace $W \subset \mathbb{R}^n$ is:

- 0 if $W = \{\vec{0}\}$ is trivial.
- the # of vectors in a basis for W, if W is non-trivial.

Example

$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^3$$

$$\Rightarrow \dim W = 2$$

$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 14 \end{pmatrix} \right\} \subset \mathbb{R}^3$$

$$\Rightarrow \dim W = 1$$

$\boxed{\text{Ex}}$

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 5 & -1 & -3 \\ -1 & -1 & -3 & 4 & 7 \end{bmatrix} \rightarrow \cdots \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -7 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

dim image $A = 2$ = # of pivot columns

dim ker $A = 3$ = # of free variables (# of non-pivot columns)

<u>Def</u> : Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, then:

- the <u>rank</u> of T is the dimension of the image of T.
- The <u>nullity</u> of T is the dimension of the kernel of T.

<u>Rmk</u> - If A is the matrix for T, then we set:

- rank A = rank T
- nullity A = nullity T

<u>Rank Theorem</u> : If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear then:

$$(\text{rank } T) + (\text{nullity } T) = \text{dimension of the domain of } T = n.$$

$\boxed{\text{Ex}}$

Let $\vec{v} \in \mathbb{R}^n$ be a non-zero vector. Let L span $\{\vec{v}\}$

What are the rank and nullity of $proj_L$?

Rank: 1 because image of $proj_L$ = Line

Nullity: $n-1$

$\boxed{\text{Ex}}$

$W = span(\beta)$ where $\beta = \{\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\vec{v}_2}\}$

Is $\vec{w} = \begin{pmatrix}0\\0\\-1\end{pmatrix} \in W$?

$$\vec{w} = 1 \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} + (-1)\begin{pmatrix}1\\0\\1\end{pmatrix} \in W$$

The systematic way to solve this:

$\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2$?

Solve $\left[\vec{v}_1, \vec{v}_2 \mid \vec{w}\right]$

Call $\begin{bmatrix}-1\\1\end{bmatrix}$ the coordinates of $\vec{w}$ relative to basis β.

Suppose $\vec{v} \in W$ has β coordinates:

$[\vec{v}]_\beta = \begin{bmatrix}7\\10\end{bmatrix}$. Whats $\vec{v}$?

Ans $\vec{v} = 7\begin{bmatrix}1\\1\\0\end{bmatrix} + 10\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}17\\7\\10\end{bmatrix}$

<u>Prop</u> : If $\beta = \{\vec{v}_1, \dots, \vec{v}_K\}$ is a basis for a subspace $W \subseteq \mathbb{R}^n$ and $\vec{v} \in W$ is some other vec, then $\exists$ unique scalers $x_1, \dots, x_K \in \mathbb{R}$ such that

$$\vec{v} = x_1 \vec{v}_1 + \dots + x_K \vec{v}_K$$

i.e $$[\vec{v}]_\beta = \begin{bmatrix} x_1 \\ \vdots \\ x_K \end{bmatrix}$$